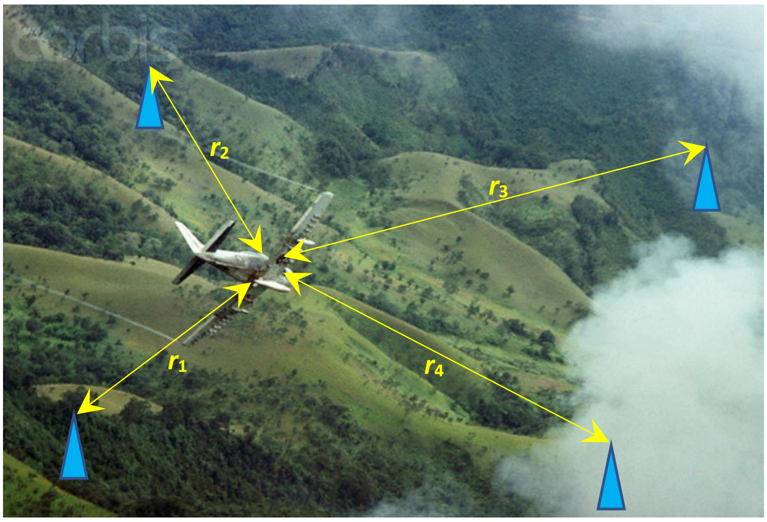
Position Fix using Range Measurements

Navigation is the determination of the position of a vehicle in space. Most navigation techniques rely on a network of identifiable landmarks, some sensors that provide measurements relating these landmarks to the vehicle, and some computations. We examine the fundamental process of position determination using range measurements.



Let there be a right-handed coordinate system fixed to a point on earth. There are N ($N = 1, 2, \dots$) antennas of *known* positions, represented as $[x_i, y_i, , z_i,]^T, i = 1, 2, \dots, N$, in this coordinate system. A sensor on the vehicle can measure the range to each antenna, giving $r_i, i = 1, 2, \dots, N$.

1. For a single ($N = 1$) antenna range, which of the following statements are true?
   a. The vehicle's position is completely unrelatable to any of the antenna's positions.
   b. The vehicle will be on the surface of a sphere centred at $[x_1, y_1, z_1]^T$ with radius r_1.
   c. The vehicle will be in the volume enclosed by two spheres, both centred at $[x_1, y_1, z_1]^T$, one with radius $r_1 + \epsilon$, the other with radius $r_1 - \epsilon$, where ϵ is the range measurement error.
   d. The vehicle will be in the *interior* of a sphere centred at $[x_1, y_1, , z_1]^T$ with radius $r_1 - \epsilon$, where ϵ is the range measurement error.
   e. The vehicle will be in the *exterior* of a sphere centred at $[x_1, y_1, , z_1]^T$ with radius $r_1 + \epsilon$, where ϵ is the range measurement error.
   f. The vehicle will be within a volume of size $8/3\pi\epsilon(3r_1^2 + \epsilon^2)$, where ϵ is the range measurement error.
   g. The vehicle will be within a volume of size $4/3\pi\epsilon^3$, where ϵ is the range measurement error.

2. For two $(N = 2)$ antenna ranges, which of the following statements are true?
    a. The vehicle's position is completely unrelatable to any of the antenna's positions.
    b. The vehicle will be on the circle defined by the intersection of two spheres, one centred at $[x_1, y_1,, z_1]^T$ with radius r_1, and the other centred at $[x_2, y_2,, z_2]^T$ with radius r_2.
    c. The vehicle will be on one of two points defined by the intersection of three (3x) spheres, each centred at $[x_i, y_i,, z_i]^T$, at radius r_i, for $i = 1, 2, 3$.
    d. With measurement errors ϵ, the vehicle will be in the volume enclosed by a "ring". This "ring" is defined by the intersection of two volumes. One volume is bounded by two spheres centred at $[x_1, y_1, z_1]^T$ of radii $r_1 + \epsilon$ and $r_1 - \epsilon$. The other volume is bounded by two spheres centred at $[x_2, y_2, z_2]^T$ of radii $r_2 + \epsilon$ and $r_2 - \epsilon$.
    e. There would be no solutions when $(r_1 + r_2)^2 < (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$.
    f. There would be no solutions when $r_1^2 + r_2^2 < (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$.
    g. The vehicle is most susceptible to the "no solutions" condition when it is on the straight line joining antennas 1 and 2, when measurement errors are present.
    h. The vehicle is most susceptible to the "no solutions" condition when it is much farther than the distance between antennas 1 and 2, when measurement errors are present. I.e., when:
    $$r_1, r_2 \gg \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$
    i. Small measurement errors can cause large position errors when the vehicle is much farther than the distance between antennas 1 and 2, when measurement errors are present. I.e., when:
    $$r_1, r_2 \gg \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

3. For three $(N = 3)$ antenna ranges, which of the following statements are true?
   a. Three (3x) is the minimum number of ranges needed to obtain a position fix in three-dimensional (3D) space.
   b. The vehicle will be on one of two points defined by the intersection of three (3x) spheres, each centred at $[x_i, y_i, , z_i]^T$, at radius r_i, for $i = 1, 2, 3$.
   c. Pythagoras' Theorem relates the *known* antenna positions $[x_i, y_i, z_i]^T, i = 1, 2, 3,$ the measured antenna ranges (r_1, r_2, r_3), and the *unknown* vehicle position $[x, y, z]^T$, to give the follow equations:

$$\begin{aligned} (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 &= r_1^2, \\ (x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2 &= r_2^2, \\ (x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2 &= r_3^2. \end{aligned} \tag{1}$$

   d. The above three *quadratic* equations in (1) can be reduced to the system of *linear* equations:

$$Au = v, \tag{2}$$

   where

$$A = \begin{bmatrix} (x_1 - x_2) & (y_1 - y_2) & (z_1 - z_2) \\ (x_2 - x_3) & (y_2 - y_3) & (z_2 - z_3) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \end{bmatrix}, \quad u = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$v = \frac{1}{2} \begin{bmatrix} x_1^2 - x_2^2 + y_1^2 - y_2^2 + z_1^2 - z_2^2 - r_1^2 + r_2^2 \\ x_2^2 - x_3^2 + y_2^2 - y_3^2 + z_2^2 - z_3^2 - r_2^2 + r_3^2 \\ x_3^2 - x_1^2 + y_3^2 - y_1^2 + z_3^2 - z_1^2 - r_3^2 + r_1^2 \end{bmatrix}.$$

   e. The vehicle's position can be obtained by solving (1).
   f. The matrix is A in (2) is full rank, and hence invertible.
   g. The vehicle's position can be obtained by solving (2).
   h. Solving the system of *quadratic* equations (1) produces two solutions.
   i. Solving the system of *linear* equations (2) produces two solutions.
   j. Closed form solutions are simple, and can be solved "by hand".
   k. Closed form solutions can be obtained by symbolic software tools like Mathematica or Matlab Symbolic Math Toolbox.